Numerous calculations have shown that the error of the solution usually does not exceed the error in specification of the boundary conditions over a wide range of variation of the latter. This testifies to the effectiveness of the solution method.

## LITERATURE CITED


1. N. I. Nikitenko, Study of Nonstationary Heat- and Mass-Transfer Processes by the Grid Method [in Russian], Naukova Dumka, Kiev (1971).
2. B. A. Boley and J. H. Weiner, Theory of Thermal Stresses, Wiley (1960).


# OPTIMIZATION OF MULTILAYER THERMAL INSULATION


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An iteration method is developed for determination of the thicknesses of layers of a multilayer thermal insulation with minimum mass, with consideration of temperature limitations. The penalty function method is employed.


Coating of surfaces by layers of thermal insulation is a widespread method of protecting thermally stressed construction details from the direct action of a high-temperature medium. One must then select the most rational variant of insulation, i.e., optimize the insulation. Often the mass of the insulating material can be considered as the optimization criterion.

We will consider the problem of heating of a multilayer thermal insulation, consisting of n layers of various materials of thickness $h_i$, i = 1, 2, ..., n. Thermal contact between layers will be assumed ideal:

$$C^i(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial y}\left(\lambda^i(T)\frac{\partial T}{\partial y}\right),$$

$$Y_{i-1} < y < Y_i, \quad 0 < t \leqslant t_c, \quad i = 1, 2, \ldots, n, \tag{1}$$

$$T(y, 0) = \varphi(y), \quad Y_0 \leqslant y \leqslant Y_n, \tag{2}$$

$$-\lambda^1(T)\frac{\partial T(Y_0, t)}{\partial y} = q_0(t), \quad t > 0, \tag{3}$$

$$-\lambda^n(T)\frac{\partial T(Y_n, t)}{\partial y} = q_n(t), \quad t > 0, \tag{4}$$

$$T(Y_i - 0, t) = T(Y_i + 0, t), \quad i = 1, 2, \ldots, n-1, \quad t > 0, \tag{5}$$

$$\lambda^i(T)\frac{\partial T(Y_i - 0, t)}{\partial y} = \lambda^{i+1}(T)\frac{\partial T(Y_i + 0, t)}{\partial y},$$

$$i = 1, 2, \ldots, n-1, \quad t > 0, \tag{6}$$

where $C^i(T)$, $\lambda^i(T)$, $\varphi(y)$, $q_0(t)$, $q_n(t)$ are known functions.

It is necessary to determine the layer thicknesses $h_i = Y_i - Y_{i-1}$, $i = 1, 2, \ldots, n$, which minimize the mass of the thermal insulation with consideration of temperature limitations in the seams between the layers. Thus, it is necessary to find the minimum of the function

$$M(\bar{h}) = \sum_{i=1}^{n} \rho_i h_i \tag{7}$$

given Eqs. (1)-(6) and the limitations


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$$T(Y_i, t) \leqslant T^i_{\max}, \quad i = 1, 2, \ldots, n, \ t > 0, \tag{8}$$

$$h_i > 0, \quad i = 1, 2, \ldots, n, \tag{9}$$

where $T^i_{\max}$ is the maximum admissible temperature in the interlayer seams.

The introduction of a penalty function transforms the original problem of finding the minimum of Eq. (7) with limitations in the form of inequality (8) to a sequence of minimization problems without limitations. Such an approach permits us to evaluate the role of the limitations and to use methods developed for solution of classical optimization problems without limitations. Of course there are difficulties in using penalty functions. In particular, the search problem is greatly complicated by the fact that the second derivatives of the penalty functions are discontinuous on the boundary of the admissible region, with the size of the discontinuities increasing with increase in the penalty parameter. To avoid this difficulty we use for the penalty function the function [1]

$$\Phi(\bar{h}, \bar{r}, \bar{\theta}) = \frac{1}{2}\sum_{i=1}^{n} r_i (g_i - \theta_i)^2_- , \tag{10}$$

where $r_i$, $\theta_i$ are penalty parameters; $g_i \geqslant 0$ are limitation functions;

$$f_- = \min(f, 0) = \begin{cases} f, & \text{if } f < 0, \\ 0, & \text{if } f \geqslant 0. \end{cases}$$

At sufficiently high values of the parameters $r_i$ the iteration process

$$\theta_i^{k+1} = \theta_i^k - \min(g_i^k, \ \theta_i^k), \quad i = 1, 2, \ldots, n, \tag{11}$$

ensures convergence of $g_i$ to zero at a geometric progression rate [1]. If the convergence rate is too slow, the possibility of increasing the parameters $r_i$ has been provided. Since the parameters $r_i$ remain limited and the parameters $\theta_i$ are varied, the surfaces upon which the second derivatives are discontinuous are removed from the solution, and the values of the discontinuities are finite.

Thus, the problem of conditional minimization is reduced to a sequence of unconditional minimization problems for the transformed function

$$P(\bar{h}, \bar{r}^k, \bar{\theta}^k) = M(\bar{h}) + \Phi(\bar{h}, \bar{r}^k, \bar{\theta}^k). \tag{12}$$

In determining the gradient of Eq. (12) calculation of the gradient of the target function (7) presents no difficulties, and a formula can be obtained for the gradient of penalty function (10) on the basis of a solution of the boundary problem conjugate to the heating problem.

Introducing for each i-th layer the dimensionless variable $\xi = (y - Y_{i-1})/h_i$, we obtain the explicit dependence of conditions (1)-(6) upon the unknown parameters $h_i$:

$$C^i(T)\frac{\partial T^i}{\partial t} = \frac{1}{h_i^2}\frac{\partial}{\partial \xi}\left(\lambda^i(T)\frac{\partial T^i}{\partial \xi}\right), \tag{13}$$

$$0 < \xi < 1, \quad 0 < t \leqslant t_c, \quad i = 1, 2, \ldots, n, \tag{14}$$

$$T^i(\xi, 0) = \varphi_i(\xi), \quad 0 \leqslant \xi \leqslant 1, \quad i = 1, 2, \ldots, n,$$

$$-\frac{\lambda^1(T)}{h_1}\frac{\partial T^1(0, t)}{\partial \xi} = q_0(t), \quad t < 0, \tag{15}$$

$$\frac{-\lambda^n(T)}{h_n}\frac{\partial T^n(1, t)}{\partial \xi} = q_n(t), \quad t > 0, \tag{16}$$

$$T^i(1, t) = T^{i+1}(0, t), \ i = 1, 2, \ldots, n-1, \ t > 0 \tag{17}$$

$$\frac{\lambda^i(T)}{h_i}\frac{\partial T^i(1, t)}{\partial \xi} = \frac{\lambda^{i+1}(T)}{h_{i+1}}\frac{\partial T^{i+1}(0, t)}{\partial \xi},$$

$$i = 1, 2, \ldots, n-1, \ t > 0, \tag{18}$$

$$T^i(1, t) \leqslant T^i_{\max}, \ i = 1, 2, \ldots, n. \tag{19}$$

For the limitation functions $g_i$ we use functions of the form

$$g_i = \int_0^{t_c} [T^i_{max} - T^i(1, t)]_- dt, \quad i = 1, 2, \ldots, n. \tag{20}$$

Considering the layer thicknesses $h_i$ as control parameters which minimize Eq. (10), following [2], we can write the conditions of the problem conjugate to Eqs. (13)-(18):

$$\frac{-\partial \psi^i}{\partial t} = \frac{\partial^2}{\partial \xi^2}(A^i \psi^i) - \frac{\partial}{\partial \xi}(B^i \psi^i) + D^i \psi^i, \tag{21}$$

$$0 < \xi < 1, \quad 0 \leqslant t < t_c, \quad i = 1, 2, \ldots, n, \tag{22}$$

$$\psi^i(\xi, t_c) = 0, \quad 0 \leqslant \xi \leqslant 1, \quad i = 1, 2, \ldots, n,$$

$$\left[\frac{\partial \lambda^1(0, t)}{\partial \xi} \frac{A^1(0, t)}{\lambda^1(0, t)} - B^1(0, t)\right] \psi^1(0, t) + \frac{\partial}{\partial \xi}[A^1(0, t)\psi^1(0, t)] = 0, \quad t < t_c, \tag{23}$$

$$\left[\frac{\partial \lambda^n(1, t)}{\partial \xi} \frac{A^n(1, t)}{\lambda^n(1, t)} - B^n(1, t)\right] \psi^n(1, t) + \frac{\partial}{\partial \xi}[A^n(1, t)\psi^n(1, t)] = -r_n(g_n - \theta_n)_-, \quad t < t_c, \tag{24}$$

$$\frac{h_i}{\lambda^i(1, t)} A^i(1, t)\psi^i(1, t) = \frac{h_{i+1}}{\lambda^{i+1}(0, t)} A^{i+1}(0, t)\psi^{i+1}(0, t), \tag{25}$$

$$i = 1, 2, \ldots, n-1, \quad t < t_c,$$

$$\left[\frac{\partial \lambda^i(1, t)}{\partial \xi} \frac{A^i(1, t)}{\lambda^i(1, t)} - B^i(1, t)\right] \psi^i(1, t) + \frac{\partial}{\partial \xi}[A^i(1, t)\psi^i(1, t)] - \left[\frac{\partial \lambda^{i+1}(0, t)}{\partial \xi} \frac{A^{i+1}(0, t)}{\lambda^{i+1}(0, t)} - B^{i+1}(0, t)\right] \psi^{i+1}(0, t) - \frac{\partial}{\partial \xi}\left[A^{i+1}(0, t)\psi^{i+1}(0, t)\right] = -r_i(g_i - \theta_i)_-, \tag{26}$$

$$i = 1, 2, \ldots, n-1, \quad t < t_c,$$

where

$$A^i(\xi, t) = \lambda^i(\xi, t)/[C^i(\xi, t) h_i^2];$$

$$B^i(\xi, t) = \frac{2}{C^i(\xi, t) h_i^2} \frac{\partial \lambda^i(\xi, t)}{\partial \xi};$$

$$D^i(\xi, t) = \left[\frac{\partial^2 \lambda^i(\xi, t)}{\partial \xi^2} - \frac{\partial C^i(\xi, t)}{\partial t} h_i^2\right] / [C^i(\xi, t) h_i^2],$$

and also a formula for calculating the gradient [3] of the penalty function

$$\frac{\partial \Phi}{\partial h_i} = \int_0^{t_c} \frac{A^i(0, t)\psi^i(0, t)}{h_i} \frac{\partial T^i(0, t)}{\partial \xi} dt - \int_0^{t_c} \frac{A^i(1, t)\psi^i(1, t)}{h_i} \frac{\partial T^i(1, t)}{\partial \xi} q_i(t) dt + \int_0^{t_c}\int_0^1 E^i(\xi, t)\psi^i(\xi, t) d\xi dt, \quad i = 1, 2, \ldots, n, \tag{27}$$

where

$$E^i(\xi, t) = -\frac{2}{h_i} \frac{\partial T^i(\xi, t)}{\partial t}.$$

The coefficients appearing in the conditions of the conjugate boundary problem Eqs. (21)-(26) and in the gradient formula Eq. (27) are determined by solution of the heating problem Eqs. (13)-(18).

The local minimum of the transformed Eq. (12) at fixed values of the parameters $\bar{r}^k$ and $\bar{\theta}^k$ is sought by the method of conjugate gradients with the following formulas [4]:

$$\bar{h}^{l+1} = \bar{h}^l - \alpha_l \bar{S}^l, \quad l = 0, 1, 2, \ldots, \tag{28}$$

TABLE 1. Effect of Thermophysical Characteristics on Mass of Two-Layer Wall

| Wall parameter | Reference solution | Optimization result | | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| $\lambda^1$ | 1 | 1 | 1 | 0,5 |
| $\lambda^2$ | 1 | 1 | 1 | 1 |
| $C^1$ | 1 | 1 | 1 | 1 |
| $C^2$ | 1 | 1 | 1 | 1 |
| $\rho_1$ | 1 | 1 | 1 | 1 |
| $\rho_2$ | 1 | 1 | 0,5 | 0,5 |
| $T^1(1,\ t)_{max}$ | 0,9585 | 0,9378 | 0,9585 | 0,9127 |
| $T^2(1,\ t)_{max}$ | 0,8337 | 0,8337 | 0,7278 | 0,8337 |
| $h_1$ | 0,5000 | 0,5432 | 0,3854 | 0,5403 |
| $h_2$ | 0,5000 | 0,4568 | 0,7133 | 0,3837 |
| $M(\bar{h})$ | 1,000 | 1,000 | 0,7421 | 0,7322 |

where

$$\bar{S}^l = -\bar{P}_l' + \beta_l \bar{S}^{l-1}, \quad \beta_l = \frac{(\bar{P}_l' - \bar{P}_{l-1}',\ \bar{P}_l')}{(\bar{P}_{l-1}',\ \bar{P}_{l-1}')}, \quad \beta_0 = 0;$$

$\bar{P}_l'$ is the gradient of the transformed Eq. (12), calculated by use of Eq. (27).

The coefficient $\alpha_l$, which determines the value of the step along the chosen direction $\bar{S}^l$ upon transition from $\bar{h}^l$ to $\bar{h}^{l+1}$, is found from the condition

$$\min_{\alpha} P^k (\bar{h}^l - \alpha_l \bar{S}^l, \bar{r}^k, \bar{\theta}^k).$$

The procedure for determination of $\alpha_l$ is simplified significantly if in evaluating the change in the penalty function Eq. (10) in the chosen direction $\bar{S}^l$ we use the solution of the boundary problem for temperature variations $\delta T^i(\xi,\ t),\ i = 1, 2, ..., n$. The conditions for this problem, in analogy to [2], are written in the form:

$$\frac{\partial(\delta T^i)}{\partial t} = A^i(\xi,\ t)\frac{\partial^2(\delta T^i)}{\partial \xi^2} + B^i(\xi,\ t)\frac{\partial(\delta T^i)}{\partial \xi} + D^i(\xi,\ t)\delta T^i + E^i(\xi,\ t)S_i,\ 0<\xi<1, \tag{29}$$

$$t>0, \quad i = 1,\ 2,\ \ldots,\ n,$$

$$\delta T^i(\xi,\ 0) = 0, \quad 0 \leqslant \xi \leqslant 1, \quad i = 1,\ 2,\ \ldots,\ n, \tag{30}$$

$$-\frac{\partial \lambda^1(0,\ t)}{\partial \xi}\delta T^1 - \lambda^1(0,t)\frac{\partial(\delta T^1(0,\ t))}{\partial \xi} = q_0(t)S_1,\ t>0, \tag{31}$$

$$-\frac{\partial \lambda^n(1,\ t)}{\partial \xi}\delta T^n(1,\ t) - \lambda^n(1,\ t)\frac{\partial(\delta T^n(1,\ t))}{\partial \xi} = q_n(t)S_n,\ t>0, \tag{32}$$

$$\delta T^i(1,\ t) = \delta T^{i+1}(0,\ t),\ i = 1,\ 2,\ \ldots,\ n-1,\ t>0, \tag{33}$$

$$h_{i+1}\left[\frac{\partial \lambda^i(1,\ t)}{\partial \xi}\delta T^i(1,\ t) + \lambda^i(1,\ t)\frac{\partial(\delta T^i(1,\ t))}{\partial \xi} - \frac{\lambda^i(1,\ t)}{h_i}\frac{\partial T^i(1,\ t)}{\partial \xi}S_i\right] = h_i\left[\frac{\partial \lambda^{i+1}(0,\ t)}{\partial \xi}\times\right. \tag{34}$$

$$\left.\times\ \delta T^{i+1}(0,\ t) + \lambda^{i+1}(0,\ t)\frac{\partial(\delta T^{i+1}(0,\ t))}{\partial \xi} - \frac{\lambda^{i+1}(0,\ t)}{h_{i+1}}\frac{\partial T^{i+1}(0,\ t)}{\partial \xi}S_{i+1}\right],$$

where $S_i$ is the change in thickness of the i-th layer.

Numerical calculations were performed for a number of examples by the method described. In doing this, the boundary problems of Eqs. (13)-(18), (21)-(26), (29)-(34) were solved using implicit difference methods.

Table 1 presents the results of the solution for the case of a two-layer wall with thermophysical characteristics independent of temperature and the following initial data: $n=2$; $q_0(t)=1$; $q_2(t)=0$; $t_k=1$; $\varphi_i(\xi)=0$; $i=1,2$. The admissible temperature values for the calculation were obtained by solving the thermal conductivity problem at known layer thickness ($h_1 = h_2 = 0.5$) [5]:

$$T^1_{max} = 0.9585, \quad T^2_{max} = 0.8337.$$

The iteration process was begun at initial layer thicknesses $h_{10} = h_{20} = 0.05$ and concluded upon fulfillment of the condition

$$|M^k - M^{k-1}| \leqslant M^k \cdot 10^{-4}.$$

Analysis of the calculations performed permits the conclusion that the method is an effective one, providing a significant reduction in computation time for both the gradient of the function $P(\bar{h}, \bar{r}, \bar{\theta})$ and for determination of the step value in the chosen direction.

## NOTATION

$M(\bar{h})$, target function; $h_i$, thickness of the i-th layer; $\rho_i$, density of material in i-th layer; n, number of layers of thermal insulation; y, spatial coordinate; t, time; $Y_i$, i = 0, 1, 2, ..., n, coordinates of layer boundaries; $C^i(T)$, volume heat capacity of material in i-th layer; $\lambda^i(T)$, thermal conductivity coefficient of material in i-th layer; $\varphi(y)$, initial temperature distribution; q, thermal flux; $t_c$, right-hand value of time interval; $T^i_{max}$, i = 1, 2, ..., n, maximum admissible temperatures on i-th boundary; $\Phi(\bar{h}, \bar{r}, \bar{\theta})$, penalty function; $\bar{r}$, $\bar{\theta}$, penalty parameters; $g_i$, function considering temperature limitations; $P(\bar{h}, \bar{r}, \bar{\theta})$, transformed function; k, number of successive unconditional minimization problem; $l$, number of iteration in search for local minimum; $\alpha$, $\beta$, $\bar{S}$, parameters of conjugate gradient method.

## LITERATURE CITED

1. F. Hill and W. Murray, Numerical Methods of Conditional Optimization [Russian translation], Mir, Moscow (1977).
2. O. M. Alifanov and V. V. Mikhailov, "Solution of the nonlinear converse thermal conductivity problem by the iteration method," Inzh.-Fiz. Zh., 35, No. 6, 1123-1129 (1978).
3. F. P. Vasil'ev, Lectures on Methods of Solving Extremal Problems [in Russian], Moscow State Univ. (1974).
4. E. Polak, Computational Methods in Optimization Methods, Academic Press (1971).
5. A. V. Lykov, Thermal Conductivity Theory [in Russian], Vysshaya Shkola, Moscow (1967).